Probability Distribution Type for the Accumulated Damage from Miner's Rule in Fatigue Design

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Abstract. Variability is present within the stress-life (S-N) fatigue analysis process. This variability propagates through the analysis process into the accumulated damage computed using Miner's Rule. This paper aims to characterise the probability distribution type of the accumulated damage from Miner's rule when accounting for variability in fatigue design parameters using a case study. Whilst the distribution type could not be conclusively selected, considerations regarding the future application of probabilistic methods for fatigue design are presented.

Keywords: Probabilistic Fatigue, Miner's Rule, Skewed Distribution.

1 Introduction

Within the sector of fatigue design, many metallic components are designed to mitigate fatigue failure using 'classical' analysis approaches based upon stress-life (S-N) curves and Miner's Rule. This approach is known as the safe-life fatigue analysis process and is currently used for aircraft landing gear [1], along with components from many other industries. The component 'safe-life' represents the number of applied cycles after which the component must be removed from service. However, the safe-life fatigue analysis process contains many sources of variability within fatigue design parameters, such as materials data, loading and component dimensions [2]. This variability propagates through the process, resulting in significant variation in the accumulated fatigue damage from Miner's Rule and hence the component safe-life. Current research work by the authors aims to develop a probabilistic approach that will compute the probability of failure (P_f) associated with a component safe-life, to better represent the statistical nature of fatigue. A probabilistic approach would model design parameters as probability distributions (e.g. *Normal*, *Weibull*, etc.) and would use probabilistic methods to propagate the variability through to the accumulated damage, enabling the computation of P_f [3]. As the accuracy of the damage probability distribution is vital for producing an accurate value of P_f, this paper aims to characterise the probability distribution type of the accumulated damage from Miner's Rule using a case study.

1.1 Stress-Life Fatigue Analysis Process

Stress-Life (S-N) fatigue analysis is a classical approach to fatigue analysis [2]. The loads applied to the component are "blocked" into 'i' blocks defined by the maximum and minimum load-levels (P_{max_i} , P_{min_i}) and the number of times a pair of load levels is applied (n_i). The loads are converted into stress-levels and subsequently into cyclic stress amplitudes (σ_{a_i}) and their associated mean stresses (σ_{m_i}). To convert the stress cycles into equivalent fully-reversed (i.e. σ_{m_i} = 0) stress amplitudes (σ_{s_i}) a model, such as the Goodman correction, is applied [2]. S-N curves represent how the number of cycles to failure (N_f) varies with the applied cyclic stress of a material. S-N curves are typically based on fully-reversed testing of material coupons. Miner's Rule can then be used to compute the fatigue damage accumulated (d_i) for a cyclic load 'i' as shown in Equation 1 [2]. The total accumulated fatigue damage (D_T) is computed by the summation of the individual damages [2] and failure is assumed to occur when D_T = 1 [2].

$$D_T = \sum d_i = \sum \frac{n_i}{N_{f_i}} = \sum \frac{Number\ of\ times\ cyclic\ stress\ `i`\ applied}{Number\ of\ cycles\ to\ failure\ for\ cyclic\ stress\ `i`\ from\ S{-}N\ curve} \quad (1)$$

1.2 Probabilistic Methods: Monte Carlo Simulation

A Monte Carlo Simulation (MCS) is a probabilistic method that performs often thousands of evaluations of a process or model, each time randomly sampling different values from the input variables, which are modelled using probability distributions [3]. This results in many values being generated for each individual output of the process. The values for each output can also be statistically characterised using a probability distribution. In the context of this paper, the input variables are the fatigue design parameters (see Section 2.1), the process/model is the S-N analysis process, and the output value is the total accumulated damage from Miner's Rule 'D_T'.

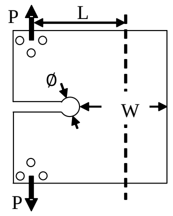
1.3 Previous Literature on the Probability Distribution Type for the Accumulated Fatigue Damage from Miner's Rule

Previous studies within the literature have proposed the *Normal* [4], *Log-Normal* [5], *3 Parameter Weibull* [6] and *Fréchet* [5] distribution types for the accumulated fatigue damage from Miner's Rule. However, these studies have relied on an assumed distribution type of 'N_f' from S-N data sets [4, 5, 6], along with an assumed S-N curve shape [5]. However, the choice of distribution type for N_f is often debated [2] and cannot be assumed *a-priori* for new S-N data sets. Previous studies have also not accounted for variability in fatigue design parameters other than N_f, such as loading and dimensional variability. To extend the work presented in previous studies, this paper aims to present a general method for generating and identifying the probability distribution type of the accumulated damage from Miner's rule, when accounting for material (S-N data), loading and dimensional variability. This objective will be achieved through the use of an MCS applied to an S-N case study. The use of an MCS means that assumptions regarding the N_f distribution type and S-N curve shape do not need to be made *a-priori*.

2 Case Study Definition

The case study geometry shown in Figure 1 [7] is the SAE Keyhole specimen, manufactured from 4340 steel [8], a typical aircraft landing gear material [1]. The hypothetical load case shown in Figure 1 was constructed to achieve a spread of stress amplitudes across the S-N curve and to ensure that stresses due to loading variability would not exceed the material Ultimate Tensile Strength (σ_{UTS}). Stress analysis equations were sourced from "*eFatigue*" [7] and a notch stress concentration factor from ref [9].

(a)



(b)

i	P_{Max_i} (kN)	P_{Min_i} (kN)	n_i	i	P_{Max_i} (kN)	P_{Min_i} (kN)	n_i
1	24.75	-11.00	50	6	22.00	-5.50	500
2	18.70	-11.00	100	7	13.75	-22.00	100000
3	22.00	-16.50	100	8	23.1	-11.00	150
4	24.75	-19.80	25	9	24.75	-16.50	50
5	13.75	-22.00	100000	10	24.75	-22.00	10

Fig. 1. (a) SAE Keyhole Geometry [7]. (b) Applied loading.

2.1 Statistical Characterisation of Fatigue Design Parameters

In order to provide inputs to the MCS, typical fatigue design parameters were statistically characterised, as shown in Table 1. Unless stated, all distributions were assumed to be *Normal*, based on the mean (μ) value and a Coefficient of Variation (CoV) defined in Equation 2, where 's' is the sample standard deviation (s values for tolerances were computed assuming the tolerance represented $\pm 3s$ as described by Haugen [3]).

$$CoV = \frac{s}{\mu} \tag{2}$$

Table 1. Statistical characterisation of case study fatigue design parameters.

Design Parameter	Statistical Characterisation
Load Levels (P_{Max_i}, P_{Min_i})	Mean load level from Figure 1 with CoV = 0.08 (typical variability of aircraft landing gear loads during touchdown [10]).
Number of Cycles (n_i)	A *discrete uniform* distribution from $0.8n_i$ to $1.2n_i$.
Nominal Width (w)	μ_w = 68.6mm with $\pm$ 0.508mm tolerance for sawing [3].
Thickness (t)	μ_t = 9.5mm with $\pm$ 0.254mm tolerance for rolled steel [3].
Hole Diameter (Ø)	$\mu_Ø$ = 9.5mm with $\pm$ 0.254mm tolerance for drilling [3].
Load Offset (L)	μ_L = 62.1mm with $\pm$ 0.381mm tolerance for hole location [3].
UTS (σ_{UTS})	μ_{UTS} = 875 MPa [8] with CoV = 0.0112 [11].

To capture variability in N_f at a given stress-level on the S-N curve, *2 Parameter (2P) Log-Normal* distributions were fitted to the coupon results at each tested stress-level in the ESDU 4340 S-N data (see Figure 2) [8]. For stress-levels that presented 'run-outs' (i.e. tests where the coupon did not fail before a predetermined N_f) [2], a constant CoV = 0.0323 was assumed as run-out data requires additional statistical methods [12] beyond the scope of this paper. The fatigue limit (σ_{FL}) was modelled as a *Normal* distribution based upon the 'Probit' method [2], giving a μ = 457MPa and s = 13. For each MCS iteration, a Probability S-N (P-S-N) curve [2] was generated based upon sampling N_f and σ_{FL} values as demonstrated in Figure 2. P-S-N curves assume a constant Probability of Survival (PoS) at all stress amplitudes and at the σ_{FL} [2].

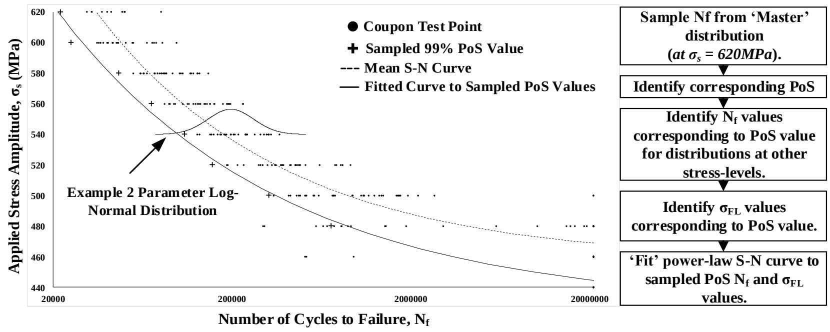


Fig. 2. Demonstration of P-S-N representation of S-N curves [2], based upon the ESDU 4340 S-N data [8]. The sampling process used to generate S-N curves for the MCS is also shown.

3 Results: Statistical Characterisation of Accumulated Damage

The MCS of the case study was repeated for 25,000 evaluations to ensure convergence of the input and output distributions. The resulting histogram of the accumulated fatigue damage from Miner's rule (D_T) is shown in Figure 3a. It can be of value to identify the distribution of the natural logarithms ($Ln(D_T)$) of the values, as shown in Figure 3b.

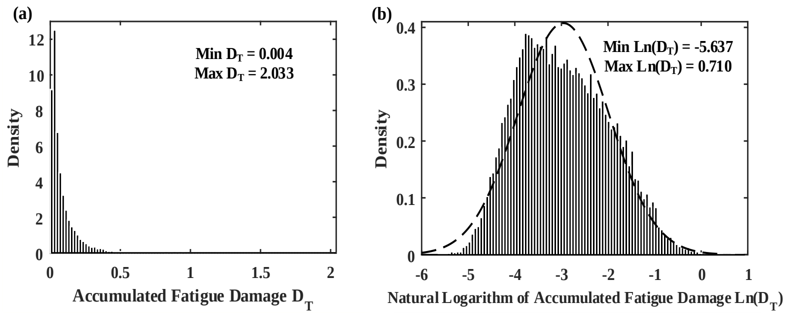


Fig. 3. Histograms of the output from the MCS. (a) shows the distribution shape of the accumulated damage from Miner's Rule (D_T) and (b) shows the distribution shape of $Ln(D_T)$ along with a Normal distribution to highlight the positive skew in $Ln(D_T)$.

As can be seen from Figure 3a, the shape of the D_T distribution is positively skewed (i.e. right-tailed). Candidate distributions capable of demonstrating positively skewed and only zero or positive values (negative damage values are not physically possible) were identified [12, 13]: *2P* and *3P Weibull, Log-Logistic, Gamma, Fréchet, Birnbaum-Saunders, Burr Type XII, Inverse Gaussian* and *Pearson Type III*. *Log-Normal* distributions were rejected due to the skew in $Ln(D_T)$ shown in Figure 3b. Maximum Likelihood Estimates (MLE) were used to 'fit' the distribution parameters (e.g. μ and s) [12]. The Cumulative Distribution Functions (CDFs) of each candidate distribution were plotted against the Empirical CDF (ECDF), which is based on the observed frequencies from the MCS [14], to visually assess the 'fit' of the candidate distributions. Only the *3P Weibull*, *Birnbaum-Saunders* and *Inverse Gaussian* provided acceptable visual fits. All other distributions showed a poor visual fit at the upper tail.

The distribution selection and MLE fitting process was repeated for the $Ln(D_T)$ values in Figure 3b. Due to the requirement for positive skew and negative values, the following candidate distributions were identified [12, 13]: *3P Log-Normal, 3P Weibull, Gumbel Maximum, Pearson Type III* and *Skew-Normal* [15]. The *Gumbel Maximum* distribution failed to provide an acceptable visual fit to the $Ln(D_T)$ ECDF.

The distribution types found to provide acceptable visual fits were then assessed for 'Goodness-of-Fit' (GoF) using the Chi-Squared (χ^2) test, which compares the frequency of observed values from the MCS results with those expected from the fitted distribution type [14]. This measure is known as the χ^2 statistic (χ^2S). This value is then compared to the critical value (χ^2C) at the 5% significance level and the candidate distribution was rejected if the computed χ^2S exceeded the χ^2C [14]. Table 2 shows the χ^2S and χ^2C for each candidate distribution. It can be seen from Table 2 that the χ^2 test rejects each of the proposed distribution types. Therefore, despite a number of the candidate distributions presenting acceptable visual fits, the distribution type for the accumulated damage from Miner's Rule could not be selected conclusively using the χ^2 test.

Table 2. χ2 test statistics and critical values for each of the candidate distributions.

Result	D_T			$Ln(D_T)$			
Distribution	*3P Weibull*	*Birnbaum Saunders*	*Inverse Gaussian*	*3P Weibull*	*3P Log-Normal*	*Pearson Type III*	*Skew Normal*
χ^2S	3478.24	841.11	405.05	981.52	852.16	919.61	849.90
χ^2C	137.70	138.81	138.81	137.70	137.70	137.70	137.70
Decision	**Reject**	**Reject**	**Reject**	**Reject**	**Reject**	**Reject**	**Reject**

3.1 Impact of Results on the Development of a Probabilistic Approach

As Table 2 has shown, the wide range of candidate distributions failed to provide an acceptable fit to the accumulated damage from Miner's Rule when using the χ^2 test. Therefore, additional distribution shapes should be considered such as *Beta* and other *Pearson* and *Burr* type distributions [13]. Regardless, the probability of failure (P_f) could still be computed from the MCS results, by identifying the number of evaluations resulting in failure. As there were 12 'failure' evaluations out of 25,000, P_f = 4.8x10^{-4}. The case study has also demonstrated that there is a pressing need for the development of a systematic process to 'down-select' the most appropriate distribution type from the wide range of candidate distributions. This is the focus of the authors' future work and could be applied to MCS results and the fatigue design parameters in Section 2.1.

To identify the cause of the skew in the $Ln(D_T)$ values, the variability in each parameter was isolated one at a time. However, the skew in the distribution persisted. This suggests that the skew is caused by a non-probabilistic element of process. This is expected to be the S-N curve, as it is the single point where all design parameters interact.

4 Conclusion

This paper has presented a Monte Carlo Simulation applied to a stress-life analysis case study with the aim of selecting the probability distribution type that best characterises the accumulated damage from Miner's rule. Whilst the distribution type could not be conclusively selected, this work has demonstrated the need to consider a wider range of distribution types than those commonly used in fatigue design, along with the need for a systematic 'down-selection process' to assist engineers in selecting distributions.

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